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Lognormal estimates of macroregional city-size distributions, 1950-1970

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Abstract. A three-parameter lognormal model is used to estimate the city-size distribution of the world and of eight UN-defined macroregions. The model is found to fit the data better than the Pareto function, and to provide a powerful means of comparing distributions among regions. Although system concentration (measured by the standard deviation index) is relatively stable in Europe and in the world at large, it is decreasing in North America, Africa, and East Asia, and increasing in Latin America and South Asia. Cities in the 250000-500000 size class are somewhat more numerous than predicted, suggesting the possibility of some kind of 'optimum'. The theory of extreme values is used to predict the most populous city of a region and to compare predictions with actual maxima, demonstrating that the largest cities in the world are well within systematic possibilities.

1 Introduction

The theory of fractal geometry as applied to spatial phenomena (Mandelbrot, 1983, chapters 29, 38) and the law of proportionate effect as applied to size distributions (Vining, 1984) are among processes which give rise to sets, the sizes of whose elements show an inverse, roughly hyperbolic association between frequency and magnitude (see Table 1 below). A large and complex collection of models has been developed to explain and describe this association as it applies to the populations of cities within regions. One possible taxonomy of these models may be found in another paper (De Cola, 1985a); Mills (1972) and Richardson (1973) give an introductory discussion and Carroll (1982) a comprehensive survey of empirical work. Dominant among the stochastic models that describe the nature of this relationship in a dynamic way are the lognormal and Pareto processes, compared by Quandt (1964) and later by Parr and Suzuki (1973) both for US data, and by Okabe (1979). It is the contention of this paper that, at least for the post World War 2 size distribution of large cities in macroregions (continents or subcontinents), the three-parameter lognormal model is more successful (figure 1) and, perhaps more important, better reflects the major descriptive dimensions of city-size distributions.

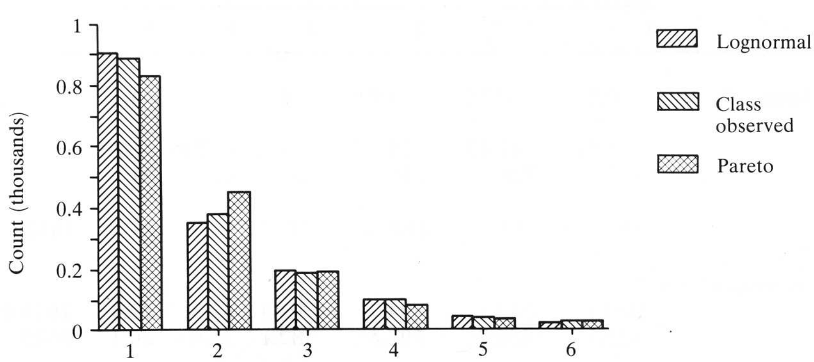

Figure 1. Predicted and observed class counts, for world cities 1970.

In the paper a theoretical development of the problem of estimating parameters from a lognormal distribution is introduced. This theory is then used in an empirical description of 1950–1970 world and macroregional city-size distributions (UN, 1980, pages 48–52) in terms of three parametric indices of extent (number of cities), scale (distributional location), and heterogeneity (distributional dispersion). Time series of these indices are next examined in the light of current theories of urban system development, and predicted maximum settlement size is also used to compare systems. The paper concludes with a summary of findings and a recommendation that regional scientists adopt the three-parameter lognormal model as the standard specification of settlement-size distributions.

2 Theory

We begin with an arbitrarily defined region containing N settlements, each with population x. Let the settlements be ordered from the smallest to the largest and indexed by i, their position in the order: $x_1, x_2, ..., x_N$. As in the present case of table 1, let the data consist of K ascending quantiles or (as they will be called) class limits: $0 < x^1 < x^2 < ... < x^K$ which are the boundaries of half-open intervals such that n^k is the cumulative class count of settlements more populous than x^k so that $N \geqslant n^1 \geqslant n^2 \geqslant ... \geqslant n^K$. The empirical distribution of x is therefore

$$P(x \leqslant x^k) = 1 - \frac{n^k}{n^1}, \qquad \text{for } x > x^1 , \tag{1}$$

where x^1 is the lower bound of the data that results from our not having a complete enumeration of the city-size distribution.

Fitting a two-parameter Pareto function to these data (Quandt, 1966) calls for finding estimates $\hat{A}$ and $\hat{b}$ of the parameters of the equation

$$\frac{n^k}{n^1} = A(x^k)^{-b} . \tag{2}$$

Such an approach usually yields satisfying results by the R^2 criterion, although a plot of residuals against x often shows a distinct inverted-U shape (Vining, 1976). Table 1 presents class count predictions for the Pareto estimation, and figure 1 compares these counts with the actual and lognormal predictions.

The major competing paradigm—the lognormal model—is somewhat more difficult to estimate because, except for the simplest fully enumerated regions, we do not

Table 1. 1970 world city-size distribution and estimates (source: UN, 1980, page 48)

	Class (k)						Total
	1	2	3	4	5	6	
Distribution							
Population lower limit (millions) (x^k)	0.1	0.25	0.5	1	2	4	
ln (x^k)	11.51	12.43	13.12	13.82	14.51	15.20	
Cumulative class count (n^k)	1615	726	345	159	63	24	
Observed class count ($n^k - n^{k+1}$)	889	381	186	96	39	24	1615
Estimates of class counts ($\hat{n}^k - \hat{n}^{k+1}$)							
Lognormal[a]	904.8	353.1	194.7	97.4	42.5	22.4	1614.9
Pareto[b]	828.1	451.2	192.5	82.1	35.0	26.1	1615

[a] Based on $\hat{\mu} = 11.677$, $\hat{\sigma} = 1.590$; $\chi^2 = 3.293$, $p = 0.655$.
[b] Based on $\hat{A} = \exp 14.56$, $\hat{b} = 1.229$; $\chi^2 = 18.59$, $p = 0.0024$.

know N, the total number of settlements. [Although it is possible to estimate N using a technique shown by De Cola (1984, pages 73-74), the Pareto approach avoids this problem entirely by examining only the right-hand tail of the distribution in equation (2).] But the three-parameter lognormal model does permit us to estimate explicitly measures of location and dispersion that fully describe the data.

Let us assume that the x are realizations of the three-parameter lognormal distribution with truncation parameter x^1, and let

$$y = \ln(x - x^1) \tag{3}$$

be a measure of x displaced by x^1. By this assumption, y is therefore a normal variate with mean μ and variance σ, so that

$$\int_{-\infty}^{y^k} (2\pi)^{-1/2} \exp\left[\frac{(\eta-\mu)^2}{2\sigma^2}\right] \mathrm{d}\eta = 1 - \frac{n^k}{n^1}, \tag{4}$$

for $k = 2, ..., K$ (Aitchison and Brown, 1957, pages 14-15). If we let Z be the inverse normal function (Abramowitz and Stegun, 1965, page 932), then

$$y^k = \ln(x^k - x^1) = \mu + \sigma Z\left(1 - \frac{n^k}{n^1}\right), \tag{5}$$

which may be estimated using least squares on the $K-1$ observations (y^k, n^k) to yield $\hat{\mu}$ and $\hat{\sigma}$ (Aitchison and Brown, 1957, pages 37-44).

Table 1 shows graphically that this technique is very fruitful; in the case of six-interval data (that is, $K = 6$) for the 1970 size distribution of 1615 (n^1) world cities more populous than $x^1 = 100000$, we obtain the estimates $\hat{\mu} = 11.677$, and $\hat{\sigma} = 1.590$ (table 2). Furthermore, the estimate of equation (5) yields $R^2 = 0.9988$. However, we are interested less in predicting values of the limits, x^k, than in predicting the counts $n^k - n^{k+1}$ which represent the essence of the data. By solving equation (5) let

$$\hat{Z}^k = \frac{1}{\hat{\sigma}}[\ln(y^k) - \hat{\mu}] \tag{6}$$

be the predicted value of the normal quantile corresponding to x^k. Then the normal probability function $\mathrm{F}(Z)$ (Abramowitz and Stegun, 1965, page 933) allows us to predict the cumulative class counts:

$$\hat{n}^k = \mathrm{n}^1[1 - \mathrm{F}(\hat{Z}^k)], \tag{7}$$

which may be used to find the (bounded) class counts. For the 1970 world distribution a comparison of the six actual and predicted class counts (table 1) yields $\chi^2 = 3.29$, for which $p = 0.65$, with five degrees of freedom, permitting the confident acceptance of the null hypothesis that the 1970 world city-size distribution is lognormal. Note as well the rather poorer predictive results of the Pareto model (figure 1).

The three-parameter lognormal model is therefore a complete specification of city-size distribution. The truncation parameter x^1 [which implies n^1 and is referred to by Malecki (1980) and by Bussière and Stovall (1981) as a 'threshold'] may be regarded as an index reflecting the spatial extent or density of the urban system; whereas μ reflects the size of a typical city or the 'scale' of the system; and σ is an index of concentration or 'heterogeneity'. Even more descriptive indices may be computed, such as the median city size,

$$\tilde{x} = x^1 + \exp\mu \;; \tag{8}$$

Table 2. Lognormal parameter estimates for regions, 1950–1970.

Year	Number of cities (n^1)	$\hat{\mu}$	$\hat{\sigma}$	χ^2	Probability
World					
1950	953	11.444	1.631	0.257	0.998
1955	1121	11.454	1.624	1.485	0.915
1960	1277	11.492	1.663	0.789	0.978
1965	1462	11.553	1.649	0.791	0.978
1970	1615	11.677	1.590	3.293	0.655
Africa					
1950	50	11.201	1.528	0.973	0.965
1955	67	11.263	1.476	2.647	0.754
1960	88	11.174	1.470	1.324	0.932
1965	107	11.386	1.424	2.702	0.746
1970	122	11.656	1.380	3.553	0.615
Latin America					
1950	71	11.406	1.767	2.267	0.811
1955	84	11.704	1.573	1.250	0.940
1960	110	11.368	1.898	4.679	0.456
1965	143	11.255	1.963	1.439	0.920
1970	173	11.385	1.855	0.598	0.988
North America					
1950	135	11.514	1.800	0.405	0.995
1955	160	11.565	1.716	0.290	0.998
1960	177	11.741	1.621	0.277	0.998
1965	185	11.958	1.661	0.120	1.000
1970	191	12.171	1.532	0.422	0.995
East Asia					
1950	147	11.465	1.707	4.823	0.438
1955	173	11.394	1.798	1.965	0.854
1960	181	11.806	1.678	0.399	0.995
1965	202	11.848	1.625	0.890	0.971
1970	213	12.103	1.529	3.961	0.555
South Asia					
1950	147	11.376	1.492	3.518	0.621
1955	171	11.381	1.502	6.143	0.293
1960	198	11.278	1.689	8.139	0.149
1965	236	11.464	1.597	2.912	0.714
1970	281	11.409	1.653	0.413	0.995
Europe					
1950	289	11.611	1.523	0.409	0.995
1955	320	11.550	1.587	1.792	0.877
1960	344	11.585	1.597	0.158	0.999
1965	374	11.591	1.581	0.374	0.996
1970	397	11.693	1.523	0.970	0.965
Oceania					
1950	10	12.239	1.267	4.187	0.523
1955	11	12.058	1.683	2.359	0.798
1960	12	11.999	1.794	0.984	0.964
1965	14	12.024	1.955	4.991	0.417
1970	15	12.069	1.841	7.262	0.202
USSR					
1950	104	11.301	1.462	5.817	0.324
1955	135	11.422	1.354	10.900	0.053
1960	167	11.381	1.401	3.545	0.617
1965	201	11.395	1.389	2.218	0.818
1970	223	11.461	1.350	1.330	0.932

the mean city size,

$$\bar{x} = x^1 + \exp(\mu + \tfrac{1}{2}\sigma^2) ; \tag{9}$$

and the estimated population of all cities (which shall be called 'total city population' in light of the threshold x^1),

$$P = n^1\bar{x} . \tag{10}$$

In spite of the concreteness of the median and mean statistics, most of the empirical discussion of the next section will be confined to references to μ and σ.

This parametric estimation is obviously an extremely powerful way to describe, summarize, and project city-size distributions for any arbitrary region (world, continent, political unit, watershed, etc). Yet there is a further step of theoretical as well as practical value. Let x_N be the population of the last (largest) city in our sample. There is some interest in comparing this value with $\hat{x}_N$, the predicted largest value, given the estimated lognormal parameters. According to Gumbel (1958, chapter 4) the lognormal distribution is of the exponential type, so that there are parameters α_N and β_N such that

$$\lim_{N \to \infty} P(y_N \leqslant \alpha_N q + \beta_N) = \exp[-\exp(-q)] , \tag{11}$$

for any q. Singpurwalla (1972, pages 709-711) presents the formulas for α_N and β_N. Although this extreme value analysis is particularly important in such engineering matters as determining the return times of floods, etc, in the present case we are interested in comparing the most probable (or modal) value of x_N with actual largest city populations. Because the distribution of y_N is only slightly negatively (!) skew, the median will suffice as an estimate:

$$p(y_N \leqslant \tilde{y}_N) \equiv \tfrac{1}{2} , \tag{12}$$

which yields

$$q = -\ln(-\ln\tfrac{1}{2}) , \qquad \tilde{y}_N = \alpha_N \ln(-\ln\tfrac{1}{2}) + \beta_N , \tag{13}$$

so that

$$\tilde{x}_N = x^1 + \exp(\tilde{y}_N) . \tag{14}$$

For the 1970 world city-size distribution this yields $\tilde{x}_N = 24\,000\,000$.

We see, therefore, that the lognormal approach to the analysis of city size allows us to arrive at an explicit parametric specification of the distribution, providing indices of the extent of the system, median city size (scale), heterogeneity (stochastic dispersion), and a prediction of the most populous city (extreme value). Certainly this is a rich harvest from an expenditure of computational energy only slightly greater than that needed to determine the much less meaningful terms of equation (2). In fact, it seems worthwhile to call for a continuation of Berry's (1961) early work and to urge the adoption of the lognormal model over the Pareto model as the standard descriptor of city-size distributions.

There are three major difficulties with the Pareto approach. First, when its parameters are estimated from the n^k city populations [as they are by Rosen and Resnick (1980)] $\hat{A}$ and $\hat{b}$ in equation (2) tend to be strongly influenced by the size of the largest cities which, being extrema though nevertheless stochastic, may tend to have what appear to be pathological values (Mandelbrot, 1965, page 332). A quantile approach to this problem is much more robust. Second, the Pareto slope b is difficult to interpret, unlike the related lognormal parameter σ, which is clearly an index of concentration related to skewness. Third, although it is possible to compute extreme order statistics for the Pareto distribution (for formulas, see Patel et al, 1976), the theory is not so well developed nor particularly useful for descriptive purposes.

The above discussion is not meant as a denial of the tremendous theoretical importance of the Pareto model. Its b parameter, for example, is clearly related to such fruitful notions as the fractal dimension of a system (Mandelbrot, 1983, chapter 38) as well as to dynamic considerations of system development (Parr, 1976; Sahal, 1978). Equation (2), moreover, is particularly valuable in estimating the far-right tail of skewed size distributions. Nevertheless, there seems to be some justification for denoting Pareto as the nonparametric counterpart of lognormal!

Table 3. Actual and predicted counts for regions, 1970.

Class (k)	Class count			Residual
	cumulative (n^k)	actual ($n^k - n^{k+1}$)	predicted ($\hat{n}^k - \hat{n}^{k+1}$)	
World				
6	24	24	22.4	1.6
5	63	39	42.5	− 3.5
4	159	96	97.4	− 1.4
3	345	186	194.7	− 8.7
2	726	381	353.1	27.9
1	1615	889	904.8	− 15.8
total		1615	1614.9	
Africa				
6	1	1	0.7	0.3
5	2	1	1.9	− 0.9
4	8	6	5.7	0.3
3	19	11	14.1	− 3.1
2	56	37	29.4	7.6
1	122	66	70.2	− 4.2
total		122	122	
Latin America				
6	4	4	3.5	0.5
5	8	4	4.0	− 0.9
4	17	9	9.7	− 0.7
3	35	18	17.7	0.3
2	69	34	31.1	2.9
1	173	104	106.1	− 2.1
total		173	173	
North America				
6	5	5	4.8	0.2
5	13	8	8.2	− 0.2
4	29	16	17.1	− 1.1
3	59	30	30.5	− 0.5
2	110	51	47.4	3.6
1	191	81	83.0	− 2.0
total		191	191	
East Asia				
6	6	6	4.7	1.3
5	11	5	8.4	− 3.4
4	31	20	18.1	1.9
3	60	29	32.9	− 3.9
2	121	61	52.6	8.4
1	213	92	96.3	− 4.3
total		213	213	

Table 3 continued

Class (k)	Class count cumulative (n^k)	actual ($n^k - n^{k+1}$)	predicted ($\hat{n}^k - \hat{n}^{k+1}$)	Residual
South Asia				
6	3	3	3.2	−0.2
5	10	7	6.0	1.0
4	23	13	13.9	−0.9
3	50	27	28.6	−1.6
2	107	57	54.9	2.1
1	281	174	174.5	−0.5
total		281	281.1	
Europe				
6	4	4	4.4	−0.4
5	15	11	9.4	1.6
4	39	24	23.0	1.0
3	82	43	48.2	−5.2
2	174	92	90.1	1.9
1	397	223	221.9	1.1
total		397	397	
Oceania				
6	0	0	0.7	−0.7
5	2	2	0.8	1.2
4	2	0	1.3	−1.3
3	6	4	2.1	1.9
2	7	1	3.1	−2.1
1	15	8	7.0	1.0
total		15	15	
USSR				
6	1	1	0.7	0.3
5	2	1	2.3	−1.3
4	10	8	7.7	0.3
3	34	24	21.3	2.7
2	82	48	50.0	−2.0
1	223	141	141.1	−0.1
total		223	223.1	

3 Estimation of 1970 world city-size distribution

Table 2 presents number of big cities, estimates of $\hat{\mu}$ and $\hat{\sigma}$, and associated χ^2 for the world and its regions in the period 1950–1970. Aitchison and Brown (1957, page 154) provide several interesting indices based on μ and σ. The coefficient of variation of the 1970 distribution is 3.40, skewness is a substantial 49.3, and kurtosis a tremendous 29000! Obviously city size is quite *abnormal*.

For the world in 1970, the median city size given by equation (8) is

$$\tilde{x} = 100000 + \exp(11.677) \approx 218000\,, \tag{15}$$

mean city size given by equation (9) is

$$\bar{x} = 100000 + \exp[11.677 + \tfrac{1}{2}(1.590)^2] \approx 517000 \tag{16}$$

and total city population given by equation (10) is

$$P = 835000000 \tag{17}$$

which is quite close to the actual value of 833000000 (UN, 1980, page 48).

We are invited to place even greater confidence in our results by the evidence of table 1, which presents in detail the evaluation of the 1970 global estimation and compares both lognormal and Pareto predictions to actual class counts. Although the Pareto predictions are seriously in error at the extreme classes (as is to be expected) only the class-2 lognormal prediction (for cities between 250000 and 500000 in population) differs by so much as 8% from the actual value. This positive residual of twenty-eight cities results in the 1970 estimation being among the worst of the entire study—and yet the χ^2 value gives scant reason to reject the null hypothesis. Furthermore, table 3 shows that, for six of the eight macroregions, residuals for class 2 are positive, suggesting that this size class may be slightly more 'successful' than other classes (this pattern holds throughout the thirty-year period). In other words, although city sizes are clearly lognormally distributed, such a frequent departure from random deviation suggests that some underlying force (other than bias in the data) is at work 'attracting' cities to this size class.

4 1950-1970 world distribution

Because the UN reported data cover the period 1950-1970, it is possible to sketch the course of big-city development during this era of post World War 2 global urbanization. Figure 2 displays changes in n^1, the number of cities, along with $\hat{\mu}$ and $\hat{\sigma}$. The first index, a count of the total number of cities more populous than 100000 in the sample, shows that about thirty-three cities per year have been matriculating to big-city status throughout the twenty-year period. The scale index, $\hat{\mu}$, on the other hand, appears to have been stable at the beginning of the period and then to have taken off around 1965: the 1950-1960 annual rate of increase was only 0.4%, whereas that of 1960-1970 was 1.6%, or four times greater. This change reflects the fact that in several major regions (Africa, North America, East Asia, and Latin America) $\hat{\mu}$ began to grow more rapidly during the 1960-1970 decade. On the other hand, the behaviour of $\hat{\sigma}$ was erratic. Global concentration may have declined during the period, which finding suggests that the world distribution of city sizes is by no means becoming more heterogeneous. Last, class 2 grew faster than its predicted size: 3.4% per year versus 3.1%, which certainly suggests that cities may remain longer at that size class than would be predicted solely by the action of the law of proportional effect.

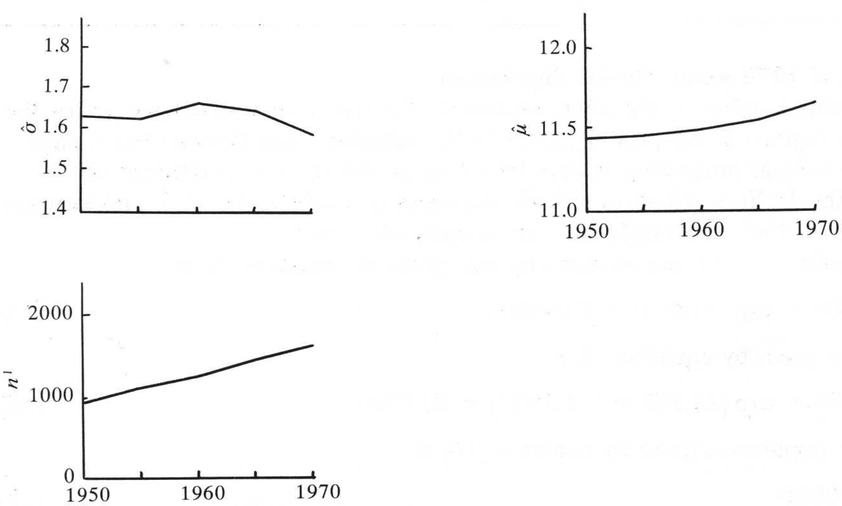

Figure 2. Time series of n^1 (number of cities), $\hat{\mu}$ (estimated mean), and $\hat{\sigma}$ (estimated variance) for world cities, 1950-1970.

The global picture is therefore one of a steadily growing number of cities whose median size is increasing at perhaps 0.7% per year but whose concentration index shows little change, at least as measured by the lognormal dispersion parameter $\hat{\sigma}$. Such regular behaviour makes it a rather straightforward matter to project total city population P for the year 2000 to roughly 1.7 billion, a number of considerable magnitude, except when compared with the figure of 6 billion which we may expect to be the world population in the year 2000 (World Bank, 1984, page 255). [Note further that this projection is substantially below that of a UN (1975) estimate for 2000 of 2.1 billion.]

5 Macroregional distributions

The UN city count data are based on an impressive listing of all of the world cities estimated to have been larger than 100000 during the period 1950-1970 (UN, 1980, pages 205-254). A worthwhile project obviously would be to examine a wide range of logical regions (continents, countries, etc) using the lognormal model to estimate parameters over time. One particularly interesting question relates to the way in which macroregional city-size distributions reflect the reginal distributions of which they are aggregations. The stochastic treatment of 'consolidated samples' is dealt with by Aitchison and Brown (1957, chapter 12) and the parametric behavior of central place systems of subsystems is explored in a paper by Beguin and Peeters (1981).

By definition, the world distribution is an aggregation of macroregional systems, which are themselves aggregations of national systems, which in turn may be aggregations of subnational systems, and so forth. My current work with African data (De Cola, 1985b) suggests that macroregional indices are quite sensitive to aggregation strategies. This sensitivity holds out for us the possibility of defining (perhaps noncoterminous) urban systems or regions in terms of descriptive stochastic indices, so that a system might be defined not exogenously in terms of political boundaries but endogenously in terms of clusters of cities that are related by the fact that they manifest a well-estimated size-distribution. This, of course, is an aspiration we have inherited from Zipf (1941). The present study, however, has had to rely on limited resources, and so is based only on those bureaucratically determined but often illogical macroreigons commonly used by the UN to organize its own data. It should therefore be kept in mind that this rich mine of urban data deserves much more detailed attention than it is receiving here.

Nevertheless, as a low-resolution examination of the global picture, the organization of the data is adequate. Figure 3 graphs the values of $\hat{\sigma}$ against $\hat{\mu}$ at 1950 and 1970 for the eight macroregions, in an attempt to show the extreme diversity of urban growth around the planet. This figure should be examined in light of the recent studies by such writers as McGreevey (1971), El-Shakhs (1972), Banks and Carr (1974), Berry and Kasarda (1977), Malecki (1980), Hall and Hay (1980), Wheaton and Shishido (1981), and De Cola (1984). These analyses suggest that economic and technological development are accompanied by a continual growth of the scale of the urban system but also by the possibility of growing and then declining urban concentration.

What first strikes one about the figure is the huge jump in $\hat{\sigma}$ for Oceania. Because this is the smallest region in terms of population and number of big cities, we should be skeptical of this result, and, indeed, the estimates for this region do show the variability typical of a small sample (table 2). It is true, nevertheless, that Oceania is a region which has only recently experienced the beginnings of big-city development. All of the other regions, however, had at least 100 cities in 1970, and so we may view their estimates with the same confidence we would place in the data themselves.

Except for Latin America and Oceania, all regions have, like the world as a whole, experienced a general increase in $\hat{\mu}$, indicating that the set of large cities is growing in scale everywhere. This growth, moreover, has been greatest in North America, East

Asia, and Africa. The first two regions are dominated by a single country where postwar urban growth has proceeded rapidly (Banerjee and Schenk, 1984), whereas in Africa, where natural increase and urban migration are presently high, median city size has grown rapidly from a low level. The USSR may be considered with the first of the above three regions to form a group of macroregions whose growth regime is, like that of the global set, marked by large increases in $\hat{\mu}$ and substantial decreases in $\hat{\sigma}$. This process reflects the continuing more rapid growth of middle-sized cities noted by the UN (1980, page 41). In other words, the city-size distribution continues to shift in a positive direction, but also to cluster.

This analysis leaves us with two exceptions: Latin America and South Asia, highly fragmented regions where big-city development is strongly a matter of increasing concentration as the largest city classes grow faster than the smaller. In fact, theory [perhaps most eloquently explicated by Alonso (1980)] suggests that political, economic, and demographic resources (power, capital, and people) will concentrate in selected subregions during development. We see this most clearly in Latin America, which in 1970 was continuing to experience increasing continental concentration, although it had achieved a level of $\hat{\sigma}$ similar to that of North America in the 1950s.

What then of Africa? In this continent, which experienced the highest rate of n^1 increase of any macroregion, extreme economic, political, and linguistic fragmentation has continued to spread urban growth among nations so that even by 1970 no 'core' nations (such as Brazil, India, or China) had arisen as foci of development. Because we know that national urban concentration has been increasing on the continent (O'Connor, 1983, chapter 2), this is clearly one case where 'consolidation' masks submacroregional changes.

With this admittedly impressionistic argument in mind, it is interesting to contrast two continents which, from the perspective perhaps of a naive visitor to Earth, might at first appear similar. North America and Africa have comparable populations and areas in terms of orders of magnitude, and by 1970 had roughly similar numbers of cities, 191 and 122, respectively. Curiously enough, the growth

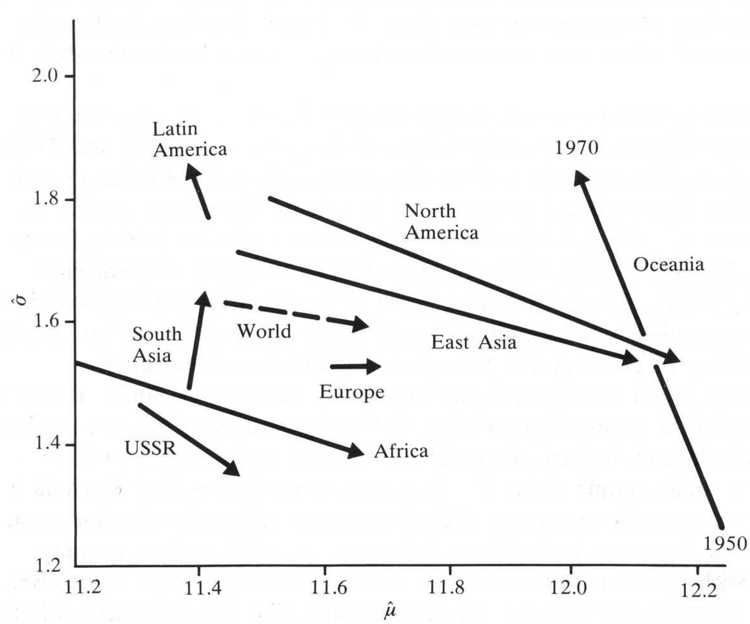

Figure 3. Estimated $\hat{\mu}$ (mean) and $\hat{\sigma}$ (variance) for regions, 1950–1970.

regimes of their urban systems were not dissimilar either: for both $\hat{\mu}$ grew (6% and 4%) while $\hat{\sigma}$ declined (−15% and −10%) as shown in figure 2. (They even have a similar number of 'states', but the analogy can be taken too far!) Yet here the similarity ends. The 191 cities of North America form an urban system in a strong sense of the word (Bourne and Simmons, 1978), a system moreover which according to Vining (1982) is experiencing regional deconcentration. The 122 cities of Africa, on the other hand, are more of a 'heap' than a system, with the cities of each nation—and frequently of each subnational region—growing in response to local forces quite powerful but limited in extent (De Cola, 1985b). It, of course, remains to be seen whether the development of an African continental urban 'system' will ever arise, even in so disjointed manner as that of Latin America.

6 Predicted maxima

This section is structured like the preceding: we shall examine the world as a whole before looking at its constituent macroregions. The population of the largest city of a region has been shown to be an extremely sensitive indicator of other important regional characteristics (De Cola, 1984). In fact, perhaps the most meaningful way to measure what is often called 'primacy' is to compare the observed population, x_N, of a first ranking city with $\hat{x}_N$, the population predicted by some reasonable model of the size distribution of which x_N is the extremum.

A dynamic way to introduce this discussion is by way of figure 4, which displays the estimated population of the world and of its largest city on a logarithmically increasing time scale for the past 7000 years, a period which, according to Calder (1983, page 164) began with the first cities. [The data are from Chandler and Fox (1974) and Clark (1977).] The explosive growth of the global maximum x_N, particularly during the last millenium, is striking. (We should note as well that N itself is always changing.) There is little evidence of settlements above a few tens of thousands (ln 10000 = 9.2) during the early Holocene, but beginning around the start of the second millenium BC, x_N began doubling roughly every 500 years until approximately AD 800, when a size of perhaps 1000000 was attained. During the Middle Ages, however, x_N was stable and may even have declined owing to cultural and epidemiological conditions—constraints which may have relevance to our future as well. The capitalist transformation of the 17th century initiated a completely new

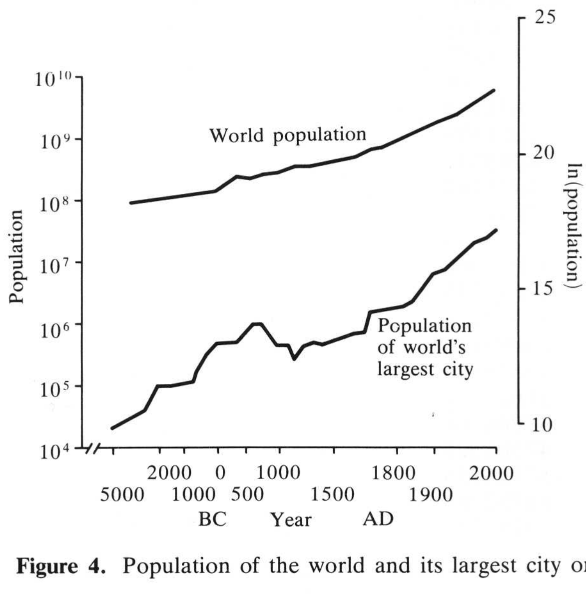

Figure 4. Population of the world and its largest city on a logarithmic time scale.

growth regime for x_N, so that for the past 300 years the population of the world's largest city (whose identity is often changing) has doubled within ever shorter intervals.

The 1970 estimate for $\tilde{x}_N = 24000000$, based on equation (14) is certainly consistent with the trend of figure 3 as is the estimated 1975 population of 19800000 for the New York–New Jersey Standard Metropolitan Statistical Area (UN, 1980, page 58), often reported as today's candidate for the world's largest city. Of course, one might just as well argue that the northeastern US megalopolis, with a 1980 population of perhaps 35000000, is the world's largest agglomeration, but at this point questions of mere scale give way to morphological puzzles.

Nevertheless, it is possible to use the 1970 world city-size distribution parameters presented above to predict $\tilde{x}_N$ for *any* value of N, as in table 4, based on equation (14). Just as the maximum flood experienced in a watershed increases with the length of the record, so does the largest city observed increase with the number of cities in a region. What is perhaps interesting, however, is that

$$\frac{\mathrm{d}\ln(x_N)^2}{\mathrm{d}^2\ln(N)} < 0\,, \tag{18}$$

and in fact, as Gumbel (1958, page 139) notes

$$\tilde{x}_N \propto [2\ln(0.4N)]^{1/2}\,, \tag{19}$$

so that the extensive development of a region (growth by adding cities) is an inefficient stimulus in itself to the extremum.

Equation (14) is used in table 5 to provide two predicted maxima for each of the macroregions: first (column A) a maximum based on world parameters $\hat{\mu}$ and $\hat{\sigma}$ applied to n^1 for each of the regions, and then (column B) a maximum based individually on the 1970 parameters for the region from table 2. Comparisons between columns

Table 4. Predicted maxima for various sample sizes (based on 1970 $\hat{\mu}$ and $\hat{\sigma}$ world parameters).

Sample size (N)	$\ln(N)$	$\ln(\tilde{x}_N)$	Maximum
1	0.0	12.29	217830
2	0.5	12.74	340060
3	1.0	13.13	506303
4	1.5	13.56	773057
7	2.0	13.96	1152596
12	2.5	14.33	1671159
20	3.0	14.67	2361576
33	3.5	15.00	3263047
55	4.0	15.30	4421764
90	4.5	15.59	5891778
148	5.0	15.86	7736030
245	5.5	16.12	10027505
403	6.0	16.37	12850536
665	6.5	16.61	16302239
1097	7.0	16.84	20494115
1808	7.5	17.06	25553821
2981	8.0	17.27	31627124
4915	8.5	17.48	38880061
8103	9.0	17.68	47501320
13360	9.5	17.87	57704858
22026	10.0	18.06	69732786
36316	10.5	18.24	83858535

A and B are another way to contrast regional city-size distributions with the world distribution: Africa and the USSR clearly have smaller-scale systems than their number of cities would predict—but for very different reasons. Comparisons between column A and the actual size of the most populous city in a region (column C) reveal that New York and Tokyo are world capitals, roughly twice as big as the number of cities in their respective macroregions would predict—and Mexico City is just big, perhaps in part because it is neither uniquely North nor Latin American. Last, columns B and C may be compared to see how each city contrasts with the maximum appropriate to its own macroregion, and we see that Cairo, like Mexico City, is a multicontinental megalopolis.

Table 5. Predicted and actual population (in millions) of larges cities

Region	Predicted maximum based on			City
	world parameters (A)	regional parameters (B)	actual maximum[a] (C)	
World	24.3	24.3	19.8	New York
Africa	7.0	4.0	6.4	Cairo
Latin America	8.4	12.7	11.9	Mexico City
North America	8.8	12.3	19.8	New York
East Asia	9.3	12.1	17.7	Tokyo
South Asia	10.8	9.8	7.8	Calcutta
Europe	12.8	10.7	10.4	London
Oceania	1.9	4.1	2.6	Sydney
USSR	9.6	4.0	7.4	Moscow

[a] Source: UN (1980, page 58) 1975 estimates.

7 Conclusion

Even though the right tails of several city-size distributions have been examined here, the lognormal model conforms to our notion of how city sizes should be distributed: a few tiny camps, a great many small settlements, many medium-sized cities, and a few great metropolises—in short, the bell curve stretched out exponentially to the right (Aitchison and Brown, 1957, page 9). The detailed empirical investigation of this paper shows that the model consistently does an excellent job of predicting world and macroregional big-city-size distributions. Regional rankings of the scale parameter $\hat{\mu}$ also reflect intuitive judgments of where largest cities are to be found.

Changes in $\hat{\sigma}$ for the world and its regions appear to indicate that urban concentration is by no means necessarily increasing, although we have much evidence that at smaller (national) scales it is often the case that the largest cities are presently attracting substantial fractions of regional population. Much of the relative stability of $\hat{\sigma}$ appears to be a result of the strong growth of the number of class-2 cities (250 000 - 500 000), whose actual counts for later years usually exceed predicted counts by significant margins.

A speculative discussion of macroregional spatial development shows that, although recent parametric time-series trends conform to theory, some regions are shown to be different from others in ways that theory has not predicted (figure 3). Latin America, for example, is manifesting increasing levels of $\hat{\sigma}$, perhaps because of growing international (intraregional) integration, and $\hat{\sigma}$ continues to decline in Africa because of persistent spatial fragmentation. Other interregional comparisons through lognormal lenses fail to contrast regions, such as North America and Africa, manifestly different in their level of spatial development.

All cities are alike in that they conform to the lognormal process, even when examined in such diverse collections as we have done here. These collections are 'fuzzy' sets in the sense used by Pousard (1977), in some instances well integrated into networks of tangible and electronic flows, and in others merely vast shapes with a feeble existence beyond the pages of atlases and the minds of politicians. But in all cases the city-size distribution can be fully described in terms of three parameters; n^1, a measure of regional extent; μ, a measure of scale; and σ, a measure of heterogeneity or concentration. When combined with a computed index of predicted extremum, $\tilde{x}_N$, we have a vivid description of the data.

It is possible dynamically to elaborate on this synoptic theory by suggesting that over time n^1 will increase linearly, eventually to approach some limit reached when (as in Europe) most available large-city sites are occupied. Scale may, however, continue to grow so long as all cities increase at roughly similar rates. And σ can be expected first to rise and then to decline as the deconcentration of advanced economic development (or underdeveloped stagnation!) lessens agglomerative economies. Finally, maximum city size x_N for any region is likely to approach some physical limit as well perhaps as a social limit imposed by the willingness of a hinterland to support it.

It is important not only to pursue theoretical elaboration of the lognormal model (Parr, 1976) but to continue in the spirit of aggressive empiricism exemplified by this paper and to extend this analysis in several directions. First, the analytical system based on computation, estimation, and evaluation must be extended in time to 1980 and pre-1950 data and in space to smaller regions, such as the city distributions compiled by Rosen and Resnick (1980) as well as those of Carroll (1980) and De Cola (1985b), in their attempts to relate concentration indices to national characteristics. It is equally important to apply these techniques to small-scale settlement systems, such as those of rural Africa (for example, see University of Ibadan, 1980) the better to understand the source of changes of which urban concentration is only the symptom.

Certainly the greatest challenge lies in our efforts to relate these parameters to the process of economic, technological, and spatial development at all scales. Never before has regional science been so close to establishing consistent and universally available data bases on which to explore models and replicate results. If we are to make continued progress in establishing concrete and shared paradigms, it will only be through the critical testing of operationally effective models on the proving ground of good data.

Note
Readers are invited to write to the author either for a listing or (by sending a disk) for MS-DOS files of the Pascal programs used to perform the analysis.

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References
Abramowitz M, Stegun I A, 1965 *Handbook of Mathematical Functions* (Dover, New York)
Aitchison J, Brown J A C, 1957 *The Lognormal Distribution* (Cambridge University Press, Cambridge)
Alonso W, 1980, "Five bell shapes in development" *Papers of the Regional Science Association* **45** 5-16
Banerjee T, Schenk S, 1984, "Lower order cities and national urbanization policies: China and India" *Environment and Planning A* **16** 487-512
Banks A S, Carr D L, 1974, "Urbanization and modernization: a longitudinal analysis" *Studies in Comparative International Development* **9** (2) 26-45
Beguin H, Peeters H I, 1981, "Urbanization in some hierarchical urban models" *Regional Science and Urban Economics* **11** 19-37

Berry B J L, 1961, "City size distributions and economic development" *Economic Development and Cultural Change* **9** 573-588
Berry B J L, Kasarda J D, 1977 *Contemporary Urban Ecology* (Macmillan, New York)
Bourne L S, Simmons J W, 1978 *Systems of Cities* (Oxford University, New York)
Bussière R, Stovall T, 1981, "A city-size distribution model based on the Weber-Fechner law of marginal effects" *Environment and Planning A* **13** 689-701
Calder N, 1983 *Timescale: an Atlas of the Fourth Dimension* (Viking Press, New York)
Carroll G R, 1980, "The chameleon and the rank-size rule: findings from a cross-national longitudinal analysis" available from the author, School of Business Administration, University of California, Berkeley, CA
Carroll G R, 1982, "National city size distributions: what do we know after 67 years of research?" *Progress in Human Geography* **6** 1-43
Chandler T, Fox G, 1974 *3000 Years of Urban Growth* (Academic Press, New York)
Clark C, 1977 *Population Growth and Land Use* (Macmillan, London)
De Cola L, 1984, "Statistical determinants of the population of a nation's largest city" *Economic Development and Cultural Change* **32** (4) 71-98
De Cola L, 1985a, "Toward a taxonomy of settlement systems models" paper presented at Annual Meeting of the Western Regional Science Association, San Diego, 27 February; copy available from the author
De Cola L, 1985b, "Urban concentration in Africa" paper presented at Annual Meeting of the Association of American Geographers, Detroit, 22 April; forthcoming in *African Urban Quarterly* **1**
El-Shakhs S, 1972, "Development, primacy and systems of cities" *Journal of Developing Areas* **7** 11-36
Gumbel E J, 1958 *Statistics of Extremes* (Columbia University Press, New York)
Hall P, Hay D, 1980 *Growth Centers in the European Urban System* (University of California Press, Berkeley, CA)
Malecki E J, 1980, "Growth and change in the analysis of rank-size distributions: empirical findings" *Environment and Planning A* **12** 41-52
Mandelbrot B, 1965, "A class of long-tailed probability distributions and the empirical distribution of city size" in *Mathematical Explorations in Behavioral Science* Eds F Massarik, P Ratoosh (R D Irwin, Homewood, IL) pp 322-332
Mandelbrot B, 1983 *The Fractal Geometry of Nature* (W H Freeman, New York)
McGreevey W P, 1971, "A statistical analysis of primacy and lognormality of Latin American cities, 1750-1920" in *The Urban Development of Latin America, 1750-1920* Ed. R Morse (Stanford University Press, Standford, CA) pp 116-219
Mills E S, 1972 *Urban Economics* (Scott-Foresman, Glenview, IL)
O'Connor A, 1983 *The African City* (Hutchison, London)
Okabe A, 1979, "An expected rank-size rule: a theoretical relationship between the rank-size rule and city size distributions" *Regional Science and Urban Economics* **9** 21-40
Parr J B, 1976, "A class of deviations from rank-size regularity" *Regional Studies* **10** 335-352
Parr J B, Suzuki T, 1973, "Settlement distributions and the lognormal distribution" *Urban Studies* **10** 335-352
Patel J K, Kapadia C H, Owen D B, 1976 *Handbook of Statistical Distributions* (Marcel Dekker, New York)
Pousard C, 1977, "Hiérarchie des places centrales et graphes Φ-flous" *Environment and Planning A* **9** 1233-1252
Quandt R E, 1964, "Statistical discrimination among alternative hypotheses and some economic regularities" *Journal of Regional Science* **5** (2) 1-23
Quandt R E, 1966, "Old and new methods of estimation and the Pareto distribution" *Metrika* **10** (1) 55-82
Richardson H W, 1973, "Theory of the distribution of city sizes: review and prospects" *Regional Studies* **7** 239-251
Rosen K, Resnick M, 1980, "The size distribution of cities, the Pareto law, and primate size" *Journal of Urban Economics* **8** 165-186
Sahal D, 1978, "A formulation of the Pareto distribution" *Environment and Planning A* **10** 1363-1376
Singpurwalla N D, 1972, "Extreme values from a lognormal law with application to air pollution problems" *Technometrics* **14** 703-711

UN, 1975 *Trends and Prospects in the Populations of Urban Agglomerations 1950–2000, as assessed in 1973–1975* (United Nations, New York)
UN, 1980 *Patterns of Urban and Rural Population Growth* (United Nations, New York)
University of Ibadan, 1980 *Report on the Resettlement Survey of the Federal Capital Territory* University of Ibadan Consultancy Services, Ibadan, Nigeria
Vining D R, 1976, "Autocorrelated growth rates and the Pareto law: a further analysis" *Journal of Political Economy* **84** 369–380
Vining D R, 1982, "Migration between the core and peripheral regions: a description and tentative explanation of the patterns in 22 countries" *Geoforum* **13** 339–410
Vining D R Jr, 1984, "The Yule–Simon model in its limiting case as a pure migration process" *Environment and Planning A* **16** 1269–1278
Wheaton W C, Shishido H, 1981, "Urban concentration, agglomeration economies, and the level of economic development" *Economic Development and Cultural Change* **30** (1) 17–30
World Bank, 1984 *World Development Report 1984* (Oxford University Press, New York)
Zipf G K, 1941 *National Unity and Disunity* (Principia Press, Bloomington, IN)